USEFUL MATHEMATICAL RELATIONSHIPS

1. Trigonometric relationships

Pythagorian relationships

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Angle sum and angle difference relationships

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

Function-product relationships

$$\sin(a)\sin(b) = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b)$$
$$\cos(a)\cos(b) = \tfrac{1}{2}\cos(a-b) + \tfrac{1}{2}\cos(a+b)$$
$$\sin(a)\cos(b) = \tfrac{1}{2}\sin(a+b) + \tfrac{1}{2}\sin(a-b)$$
$$\cos(a)\sin(b) = \tfrac{1}{2}\sin(a+b) - \tfrac{1}{2}\sin(a-b)$$

Function-sum and function-difference relationships

$$\sin(a) + \sin(b) = 2\sin[(a+b)/2]\cos[(a-b)/2]$$
$$\sin(a) - \sin(b) = 2\cos[(a+b)/2]\sin[(a-b)/2]$$
$$\cos(a) + \cos(b) = 2\cos[(a+b)/2]\cos[(a-b)/2]$$
$$\cos(a) - \cos(b) = -2\sin[(a+b)/2]\sin(a-b)/2]$$

Euler relationships

$$e^{ix} = \cos(x) + i\sin(x) \qquad\qquad i = (-1)^{1/2}$$
$$\sin(x) = (e^{ix} - e^{-ix})/2i \qquad\qquad \cos(x) = (e^{ix} - e^{-ix})/2$$

Power series expansions for sin and cosine

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \qquad \sin(kx) = kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \frac{(kx)^7}{7!} + ...$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \qquad \cos(kx) = k - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \frac{(kx)^6}{6!} + ...$$

2. Indefinite integrals

290. $\displaystyle\int (\sin ax)\, dx = -\frac{1}{a}\cos ax$

291. $\displaystyle\int (\cos ax)\, dx = \frac{1}{a}\sin ax$

296. $\displaystyle\int (\sin^2 ax)\, dx = -\frac{1}{2a}\cos ax \sin ax + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4a}\sin 2ax$

297. $\displaystyle\int (\sin^3 ax)\, dx = -\frac{1}{3a}(\cos ax)(\sin^2 ax + 2)$

298. $\displaystyle\int (\sin^4 ax)\, dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

299. $\displaystyle\int (\sin^n ax)\, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int (\sin^{n-2} ax)\, dx$

302. $\displaystyle\int (\cos^2 ax)\, dx = \frac{1}{2a}\sin ax \cos ax + \frac{1}{2}x = \frac{1}{2}x + \frac{1}{4a}\sin 2ax$

303. $\displaystyle\int (\cos^3 ax)\, dx = \frac{1}{3a}(\sin ax)(\cos^2 ax + 2)$

304. $\displaystyle\int (\cos^4 ax)\, dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

305. $\displaystyle\int (\cos^n ax)\, dx = \frac{1}{na}\cos^{n-1} ax \sin ax + \frac{n-1}{n}\int (\cos^{n-2} ax)\, dx$

316. $\displaystyle\int (\sin mx)(\sin nx)\, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$

317. $\displaystyle\int (\cos mx)(\cos nx)\, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$

318. $\displaystyle\int (\sin ax)(\cos ax)\,dx = \frac{1}{2a}\sin^2 ax$

319. $\displaystyle\int (\sin mx)(\cos nx)\,dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$

320. $\displaystyle\int (\sin^2 ax)(\cos^2 ax)\,dx = -\frac{1}{32a}\sin 4ax + \frac{x}{8}$

321. $\displaystyle\int (\sin ax)(\cos^m ax)\,dx = -\frac{\cos^{m+1} ax}{(m+1)a}$

322. $\displaystyle\int (\sin^m ax)(\cos ax)\,dx = \frac{\sin^{m+1} ax}{(m+1)a}$

389. $\displaystyle\int x(\sin ax)\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$

390. $\displaystyle\int x^2(\sin ax)\,dx = \frac{2x}{a^2}\sin ax - \frac{a^2 x^2 - 2}{a^3}\cos ax$

391. $\displaystyle\int x^3(\sin ax)\,dx = \frac{3a^2 x^2 - 6}{a^4}\sin ax - \frac{a^2 x^3 - 6x}{a^3}\cos ax$

393. $\displaystyle\int x(\cos ax)\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$

394. $\displaystyle\int x^2(\cos ax)\,dx = \frac{2x\cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3}\sin ax$

395. $\displaystyle\int x^3(\cos ax)\,dx = \frac{3a^2 x^2 - 6}{a^4}\cos ax + \frac{a^2 x^3 - 6x}{a^3}\sin ax$

399. $\displaystyle\int x(\sin^2 ax)\,dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$

400. $\displaystyle\int x^2(\sin^2 ax)\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x\cos 2ax}{4a^2}$

401. $\displaystyle\int x(\sin^3 ax)\,dx = \frac{x\cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x\cos ax}{4a} + \frac{3\sin ax}{4a^2}$

402. $\displaystyle\int x(\cos^2 ax)\,dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$

403. $\displaystyle\int x^2(\cos^2 ax)\,dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x\cos 2ax}{4a^2}$

404. $\displaystyle\int x(\cos^3 ax)\,dx = \frac{x\sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x\sin ax}{4a} + \frac{3\cos ax}{4a^2}$

3. Definite integrals

661. $\displaystyle\int_0^\infty x^n e^{-ax}\,dx = \begin{cases} \dfrac{\Gamma(n+1)}{a^{n+1}}, & (n > -1, a > 0) \\[2mm] \qquad\text{or} \\[2mm] \dfrac{n!}{a^{n+1}}, & (a > 0,\, n \text{ positive integer}) \end{cases}$

662. $\displaystyle\int_0^\infty x^n \exp(-ax^p)\,dx = \frac{\Gamma(k)}{pa^k}, \qquad \left(n > -1, p > 0, a > 0, k = \frac{n+1}{p}\right)$

663. $\displaystyle\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right), \qquad (a > 0)$

664. $\displaystyle\int_0^\infty x e^{-x^2}\,dx = \tfrac{1}{2}$

665. $\displaystyle\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$

666. $\displaystyle\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$

667. $\displaystyle\int_0^\infty x^{2n+1} e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}, \qquad (a > 0)$